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Regular Behavior of Orthogonal Polynomials and Its Localization*

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A simple proof is given for a localization theorem of H. Stahl.

Let μ denote a finite Borel measure on the real line $\mathbb{R}$ with compact support $S(\mu) := \operatorname{supp}(\mu)$. By $\Omega = \Omega(\mu)$ we denote the complement of the support $S(\mu)$, i.e., $\Omega := \bar{\mathbb{C}} \backslash S(\mu)$, and by $I(\mu)$ the smallest interval on $\mathbb{R}$ containing the support of μ.

We shall always assume that the support of μ consists of infinitely many points. Then we can form the uniquely existing orthonormal polynomials

$$p_n(\mu; z) = \gamma_n(\mu)\, z^n + \cdots, \qquad \gamma_n(\mu) > 0$$

with respect to μ

$$\int p_n(\mu; z)\, \overline{p_m(\mu; z)}\, d\mu(z) = \delta_{n,m},$$

where $\delta_{n,m} = 1$ if $n = m$ and $\delta_{n,m} = 0$ otherwise.

In what follows $\operatorname{cap}(S)$ denotes the (outer logarithmic) capacity of a bounded set $S \subseteq \mathbb{C}$; i.e., $\operatorname{cap}(S) = \inf_U \operatorname{cap}(U)$, where the infimum extends over all open sets $U \supseteq S$ (see Chapter 11, Section 2 of [1]), and we say that a property holds qu.e. (quasi everywhere) on a set $S \subseteq \mathbb{C}$ if it holds on S with possible exceptions on a subset of capacity zero. By $g_\Omega(z; \infty)$ we denote the Green function of Ω with logarithmic pole at infinity.

For the formulation of our results we introduce the following convergence notion. We say that a limit relation

$$\liminf_{n \to \infty} |f_n(z)| \geqslant h(z)$$

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holds true *locally uniformly* in an open set D if for every $z \in D$ and $z_n \to z$ as $n \to \infty$ we have

$$\liminf_{n \to \infty} |f_n(z_n)| \geqslant h(z).$$

Thus, the two notions "locally uniformly in D" and "uniformly on compact subsets of D" are different; in general the latter is stronger than the former one. However, if h is continuous, or merely upper semi-continuous (for a lim sup relation: h lower semi-continuous), then these two convergent notions coincide.

The following two results are the basis of the definition of the regularity of a measure below.

LEMMA 1 (see Section 3.9 of [4]). *For any μ the following are true*:

(i)

$$\liminf_{n \to \infty} \gamma_n(\mu)^{1/n} \geqslant \frac{1}{\operatorname{cap}(S(\mu))}. \tag{1}$$

(ii)

$$\liminf_{n \to \infty} |p_n(\mu; z)|^{1/n} \geqslant e^{g_\Omega(z; \infty)} \tag{2}$$

locally uniformly in $\mathbb{C} \setminus I(\mu)$.

(iii) *For every infinite subsequence $N \subseteq \mathbb{N}$ we have*

$$\limsup_{n \to \infty, n \in N} |p_n(\mu; z)|^{1/n} \geqslant 1 \qquad \textit{for qu.e.} \quad z \in S(\mu). \tag{3}$$

The next assertion explains when we have equality in the above estimates.

LEMMA 2 (see Theorem 1 in [4]). *The following three assertions are equivalent*:

(i) *The limit*

$$\lim_{n \to \infty} \gamma_n(\mu)^{1/n} = \frac{1}{\operatorname{cap}(S(\mu))} \tag{4}$$

holds true.

(ii) *The limit*

$$\lim_{n \to \infty} |p_n(\mu; z)|^{1/n} = e^{g_\Omega(z; \infty)} \tag{5}$$

holds true locally uniformly in $\bar{\mathbb{C}} \setminus I(\mu)$.

(iii) *The limit*

$$\limsup_{n\to\infty} |p_n(\mu; z)|^{1/n} = 1 \tag{6}$$

holds true qu.e on $S(\mu)$.

It easily follows from (5), (6), and the principle of descent [1, Theorem 1.3] that

$$\limsup_{n\to\infty} |p_n(\mu; z)|^{1/n} \leqslant e^{g_\Omega(z;\infty)}$$

holds true locally uniformly on $\mathbb{C}$. We shall use this remark in the proof below.

DEFINITION. If one of the three assertions of Lemma 2 holds true, then the orthonormal polynomials $p_n(\mu; z)$, $n \in \mathbb{N}$, associated with the measure μ are said to have *regular* (nth *root*) *asymptotic behavior*, and we write $\mu \in \mathbf{Reg}$. We shall refer to $\mu \in \mathbf{Reg}$ simply as μ is *regular*.

Orthogonal polynomials with regular behavior are the analogues of the classical orthogonal polynomials for general measures, and this notion is extremely important and useful in applications. Therefore, the following theorem of Herbert Stahl [2], which asserts the surprising fact that the regularity of a measure is basically a local property, is of fundamental importance in the theory.

THEOREM A. *Let* $K \subseteq \mathbb{R}$ *be a compact set such that the support of* $\mu_K := \mu|_K$ *is an infinite set and*

$$\operatorname{cap}(K \cap S(\mu)) = \operatorname{cap}(\operatorname{Int}(K) \cap S(\mu)) \tag{7}$$

holds, where Int *denotes the interior in* $\mathbb{R}$. *Then the following statements are equivalent.*

(i) $\mu_K \in \mathbf{Reg}$; *i.e., the sequence* $\{p_n(\mu_K; \cdot)\}_{n=0}^{\infty}$ *has regular* (nth *root*) *asymptotic behavior.*

(ii) *We have*

$$\limsup_{n\to\infty} |p_n(\mu; z)|^{1/n} \leqslant e^{g_{\Omega(\mu_K)}(z;\infty)}$$

locally uniformly for $z \in \mathbb{C}$.

(iii) *The relation*

$$\limsup_{n\to\infty} |p_n(\mu; z)|^{1/n} \leqslant 1$$

holds quasi everywhere on $S(\mu_K)$.

(iv) *For any sequence* $\{P_n\}$ *of nonzero polynomials of degree* $\deg(P_n) \leqslant n$ *we have*

$$\limsup_{n\to\infty} \left(\frac{|P_n(z)|}{\|P_n\|_{L^2(\mu_K)}}\right)^{1/n} \leqslant e^{g_{\Omega(\mu_K)}(z;\,\infty)}$$

locally uniformly for $z \in \mathbb{C}$.

(v) *For any sequence* $\{P_n\}$ *as in* (iv)

$$\limsup_{n\to\infty} \left(\frac{|P_n(z)|}{\|P_n\|_{L^2(\mu_K)}}\right)^{1/n} \leqslant 1$$

for quasi every $z \in S(\mu_K)$.

If $S(\mu_K)$ *is a regular set with respect to the solution of the Dirichlet problem in the domain* $\Omega(\mu_K)$, *then in assertion* (v) *the asymptotic inequality holds true not only quasi everywhere, but uniformly on* $S(\mu_K)$, *and in* (iii) *we have an upper inequality* $\leqslant$ *uniformly on* $S(\mu_K)$.

The main purpose of this paper is to give a simple proof for Stahl's result. This is warranted by the importance of the theorem and by the relative complexity of Stahl's proof. He used a very fine potential technique for "zero surgery" which certainly will have applications elsewhere. However, here we show that the above result can be proved rather simply.

Actually we shall prove a slight improvement of Theorem A, namely in assertions (iv) and (v) we shall replace $L^2(\mu_K)$-norms by $L^2(\mu)$-norms. With this we get more unified statements, and Stahl's version can be easily derived from our variant. Thus, we shall prove

THEOREM 1. *Let* K *be a compact set such that the support of* $\mu_K := \mu|_K$ *is an infinite set and* (7) *holds. Then the following statements are equivalent.*

(i) $\mu_K \in \mathrm{Reg}$; *i.e., the sequence* $\{p_n(\mu_K;\cdot)\}_{n=0}^{\infty}$ *has regular* (nth *root*) *asymptotic behavior.*

(ii) *We have*

$$\limsup_{n\to\infty} |p_n(\mu; z)|^{1/n} \leqslant e^{g_{\Omega(\mu_K)}(z;\,\infty)} \tag{8}$$

locally uniformly for $z \in \mathbb{C}$.

(iii) *The relation*

$$\limsup_{n\to\infty} |p_n(\mu; z)|^{1/n} \leqslant 1 \tag{9}$$

holds quasi everywhere on $S(\mu_K)$.

(iv) *For any sequence* $\{P_n\}$ *of nonzero polynomials of degree* $\deg(P_n) \leqslant n$ *we have*

$$\limsup_{n\to\infty} \left(\frac{|P_n(z)|}{\|P_n\|_{L^2(\mu)}}\right)^{1/n} \leqslant e^{g_{\Omega(\mu_K)}(z;\,\infty)} \tag{10}$$

locally uniformly for $z \in \mathbb{C}$.

(v) *For any sequence* $\{P_n\}$ *as in* (iv)

$$\limsup_{n\to\infty} \left(\frac{|P_n(z)|}{\|P_n\|_{L^2(\mu)}}\right)^{1/n} \leqslant 1 \tag{11}$$

for quasi every $z \in S(\mu_K)$.

If $S(\mu_K)$ *is a regular with respect to the solution of the Dirichlet problem in the domain* $\Omega(\mu_K)$, *then the above relations hold uniformly in the range described.*

Proof of Theorem 1. (ii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (v) are immediate if we expand the polynomials P_n in their (finite) Fourier series in the orthogonal polynomials $\{p_k(\mu;\cdot)\}$. In a similar manner, (i) $\Rightarrow$ (ii) follows from Lemma 2 by Fourier expansion of $p_n(\mu;\cdot)$ into $\{p_k(\mu_K;\cdot)\}_{k=0}^{\infty}$ (cf. also the remark made after Lemma 2). Since (iv) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is trivial, it has only remained to prove (v) $\Rightarrow$ (i). This is basically Lemma 4.2 of [2], so with the following proof we give a short proof for that lemma, as well.

Let us suppose on the contrary that (i) is false. Then, by (3) and (iii) in Lemma 2 we have

$$\limsup_{n\to\infty} |p_n(\mu_K;x)|^{1/n} > 1 \tag{12}$$

on a subset of $S(\mu_K)$ of positive capacity. Since qu.e. point of $S(\mu_K)$ is a regular boundary point with respect to the Dirichlet problem for the domain $\Omega_K := \Omega(\mu_K) = \bar{\mathbb{C}} \backslash S(\mu_K)$ (see [3, Theorem III.33]), (12) holds true at some regular point x_0. Let $N_1 \subseteq \mathbb{N}$ and $0 < \eta < 1/2$ be such that the limit

$$\lim_{n\to\infty,\, n\in N_1} |p_n(\mu_K;x_0)|^{1/n} > e^{2\eta} \tag{13}$$

exists and satisfies the stated inequality. Let $\nu_{p_n(\mu_K;\cdot)}$ be the normalized counting measure on the zeros of $p_n(\mu_K;\cdot)$; i.e. $\nu_{p_n(\mu_K;\cdot)}$ is the measure that places mass $1/n$ to each zero of $p_n(\mu_K;\cdot)$. Since these measures are supported in the smallest interval $I(\mu)$ containing the support of μ, Helly's selection theorem can be applied and we can select another subsequence $N_2 \subseteq N_1$ such that the limits

$$\lim_{n\to\infty,\, n\in N_2} \nu_{p_n(\mu_K;\cdot)} = \nu, \qquad \lim_{n\to\infty,\, n\in N_2} \gamma_n(\mu_K)^{1/n} = e^c \tag{14}$$

exist, where the first limit is taken in the weak* topology on measures with support in $I(\mu)$. Obviously, ν is a probability measure on $I(\mu)$ and $c \in \mathbb{R} \cup \{\infty\}$.

We can deduce from (13) and the principle of descent (see [1, Theorem 1.3]) for the logarithmic potential

$$p(\nu, x) := \int \log \frac{1}{|x-t|} \, d\nu(t)$$

of ν that

$$-p(\nu, x_0) + c > 2\eta = g_{\Omega_K}(x_0, \infty) + 2\eta. \tag{15}$$

Since here both sides are continuous in the fine topology (see [1, Chap. III]), we must have

$$-p(\nu, x_1) + c > g_{\Omega_K}(x_1, \infty) + 2\eta \tag{16}$$

for some $x_1 \notin \mathbb{R}$.

Let now $K_1 \subseteq K_2 \subseteq \cdots \subseteq \operatorname{Int}(K) \cap S(\mu) \subseteq S(\mu_K)$ be an increasing sequence of compact sets with

$$\lim_{m \to \infty} \operatorname{cap}(K_m) = \operatorname{cap}(\operatorname{Int}(K) \cap S(\mu)) = \operatorname{cap}(S(\mu_K)),$$

where the last equality is a consequence of (7) and

$$\operatorname{Int}(K) \cap S(\mu) \subseteq S(\mu_K) \subseteq K \cap S(\mu).$$

Since

$$g_{\mathbb{C} \setminus K_m}(z, \infty) \geq g_{\mathbb{C} \setminus S(\mu_K)}(z, \infty),$$

and

$$\lim_{m \to \infty} (g_{\mathbb{C} \setminus K_m}(z, \infty) - g_{\mathbb{C} \setminus S(\mu_K)}(z, \infty))|_{z=\infty} = \lim_{m \to \infty} \log \frac{\operatorname{cap}(S(\mu_K))}{\operatorname{cap}(K_m)} = 0,$$

it follows from Harnack's inequality that together with (16) we must also have

$$-p(\nu, x_1) + c > g_{\mathbb{C} \setminus K_m}(x_1, \infty) + 2\eta \tag{17}$$

for large m. Fix such an m. Equation (17) implies via the principle of domination (see [1, Theorem 1.27]) (recall that $g_{\mathbb{C} \setminus K_m}$ is the difference of a constant and the logarithmic potential of the equilibrium measure of K_m

and this latter one has finite logarithmic energy for large m) that we must have

$$-p(\nu, x)+c > g_{\mathbb{C}\setminus K_m}(x, \infty)+2\eta$$

on a set $E\subseteq K_m$ of positive capacity. By the lower envelope theorem ([1, Theorem 3.8]) we can then conclude

$$\limsup_{n\to\infty, n\in N_2} |p_n(\mu_K; x)|^{1/n} > e^{2\eta} \tag{18}$$

for qu.e. $x\in E$; hence we can assume (18) for all $x\in E$. Recall now that $E\subseteq K_m\subseteq \mathrm{Int}(K)\cap S(\mu)$, and $\mathrm{Int}(K)$ is the union of countably many open intervals Since $\mathrm{cap}(E)>0$, there is a subinterval $J\subseteq \mathrm{Int}(K)$ with $\mathrm{cap}(E\cap J)>0$. Hence, by changing E if necessary, we may assume besides $\mathrm{cap}(E)>0$ and (18) for every $x\in E$ that $E\subseteq[\alpha, \beta]$, where $[\alpha, \beta]$ is a proper subinterval of the open interval $J\subseteq K$.

Now we distinguish two cases.

Case I. The constant c in (14) is finite. Then the polynomials $\{p_n(\mu_K; \cdot)\}_{n\in N_2}$ cannot grow exponentially on compact subset of $\mathbb{C}$, hence there is a C such that

$$|p_n(\mu_K; x)|^{1/n}\leqslant C, \qquad x\in S(\mu), \qquad n\in N_2. \tag{19}$$

Choose a polynomial Q such that $0\leqslant Q\leqslant 1$ on $I(\mu)$,

$$e^{-\eta}\leqslant Q(x)\leqslant 1 \qquad \text{for} \quad x\in[\alpha, \beta] \tag{20}$$

and

$$0\leqslant Q(x)\leqslant 1/C \qquad \text{for} \quad x\in I(\mu)\setminus J \tag{21}$$

are satisfied, and consider the polynomials

$$P_{n(1+k)}(x)=p_n(\mu_K; x)\, Q(x)^n, \tag{22}$$

where k denotes the degree of Q.

By (19) and (21) we have for $n\in N_2$

$$\|P_{n(1+k)}\|^2_{L^2(\mu)}\leqslant \|p_n(\mu_K; \cdot)\|^2_{L^2(\mu_K)}+\mu(I(\mu)\setminus J)=O(1), \tag{23}$$

and so it follows from (18) and (20) that for every $x\in E$

$$\limsup_{n\to\infty, n\in N_2}\left(\frac{|P_{n(1+k)}(x)|}{\|P_{n(1+k)}\|_{L^2(\mu)}}\right)^{1/n(k+1)}$$
$$\geqslant \limsup_{n\to\infty}\left(\frac{e^{2n\eta}e^{-n\eta}}{O(1)}\right)^{1/n(1+k)}=e^{\eta/(1+k)}>1,$$

which shows that assertion (v) is not true because $\operatorname{cap}(E) > 0$. This proves the implication (v) $\Rightarrow$ (i) in the case $c < \infty$.

Case II. $c = \infty$. In this case the argument is very similar to the one above, only we work with the monic orthogonal polynomials

$$q_n(\mu_K; x) = \frac{1}{\gamma_n(\mu_K)} p_n(\mu_K; x).$$

In fact, by (14) and the lower envelope theorem we have

$$\limsup_{n \to \infty, n \in N_2} |q_n(\mu_K; z)|^{1/n} = \exp(-p(\nu; z)) > 0$$

for qu.e. $z \in \mathbb{C}$; hence we can choose $d > 0$ and C so that

$$\limsup_{n \to \infty} |q_n(\mu_K; x)|^{1/n} \geqslant d, \qquad x \in E^* \tag{18'}$$

for some $E^* \subseteq (\alpha, \beta)$, $\operatorname{cap}(E^*) > 0$ and

$$|q_n(\mu_K; x)|^{1/n} \leqslant C, \qquad x \in S(\mu), \qquad n \in N_2. \tag{19'}$$

are satisfied. Choose now Q according to (20) and

$$0 \leqslant Q \leqslant d/2C \qquad \text{for} \quad x \in I(\mu) \backslash J.$$

For the polynomials (22) with $p_n(\mu_K; x)$ replaced by $q_n(\mu_K; x)$ we have now like in (23)

$$\|P_{n(1+k)}\|_{L^2(\mu)} \leqslant (o(1))^n + O\left(\left(\frac{d}{2}\right)^n\right),$$

(recall that $c = \infty$) hence (18′) yields for $x \in E^*$

$$\limsup_{n \to \infty} \left(\frac{|P_{n(1+k)}(x)|}{\|P_{n(1+k)}\|_{L^2(\mu)}}\right)^{1/n(k+1)} \geqslant (2e^{-\eta})^{1/(1+k)} > 1;$$

i.e., (v) is false again.

References

1. N. S. Landkof, "Foundations of Modern Potential Theory," Grundlehren Math. Wiss. Vol. 190, Springer-Verlag, New York, 1972.
2. H. Stahl, On regular nth root asymptotic behavior of orthonormal polynomials, *J. Approx. Theory* **65** (1991), 383–419.
3. M. Tsuji, "Potential Theory in Modern Function Theory," Maruzen, Tokyo, 1959.
4. J. L. Ullman, A survey of exterior asymptotics for orthogonal polynomials associated with a finite interval and a study of the case of general weight measures, *in* "Approximation Theory and Spline Functions" (S. P. Singh *et al.*, Eds.), pp. 467–478, Reidel, Dordrecht, 1984.